Topological filters and high-pass/low-pass devices for solitons in inhomogeneous networks

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We show that, by inserting suitable finite networks at a site of a chain, it is possible to realize filters and high-pass/low-pass devices for solitons propagating along the chain. The results are presented in the framework of coupled optical waveguides; possible applications to different contexts, such as photonic lattices and Bose-Einstein condensates in optical networks are also discussed. Our results provide a first step in the control of the soliton dynamics through the network topology.

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I. INTRODUCTION

Soliton dynamics in discrete structures features remarkable effects, presenting a very high—theoretical and experimental—interest [1–3]. In particular, a very active and well-established field of research is the study of wave transmission in presence of nonlinearity on translationally invariant chains, where solitonic and breather solutions have been extensively studied [2,4]. For solitons propagating on more general discrete structures, the role played by the topology of the network (i.e., how the sites of the network are connected between them) and the interplay and competition of topology and nonlinearity in the soliton dynamics are still an open problem, and one expects that new interesting phenomena should arise. The interest on this topic is also motivated by the fact that in several systems, like networks of nonlinear waveguide arrays [3], arrays of super conducting networks [5], Bose-Einstein condensates in optical lattices [6], and silicon-based photonic crystals [7], one can, at some extent, engineer the shape (i.e., the topology) of the network. On this respect, the effects of inhomogeneity on soliton propagation and on localized modes have been investigated in Y junctions [8,9], junctions made of two infinite waveguides and waveguide couplers [10], lattices featuring topological dislocations created by the interference between plane waves and waves with nested vortices [11], and scattering through a topological perturbation [12].

In this paper we consider solitons propagating on inhomogeneous discrete structures, and we show that it is possible to use the inhomogeneity—i.e., the shape of the network—to realize filters for the soliton momentum (allowing for the propagation of solitons with a given velocity) and high-pass/low-pass devices (allowing for the propagation of solitons with high/low velocity). In particular, we will focus on the discrete nonlinear Schrödinger equation (DNLSE) on inhomogeneous networks: the DNLSE is a paradigmatic example of nonlinear model which has been extensively studied on regular lattices. Indeed it describes the properties of several real systems, like coupled waveguides arrays [3], nonlinear discrete electrical networks [13], and Bose-Einstein condensates in optical lattices [14] (we also refer to the reviews [2,15] for more references on applications of the DNLSE). In particular, soliton propagation has been experimentally observed in coupled optical waveguides [16] and these systems represent a promising physical setup for the study of the effects of topology on soliton propagation. A topological engineering of waveguides appears to be a realizable task, as illustrated by Fig. 1. Other experimental systems could be used to engineer and build inhomogeneous networks. Two-dimensional optically induced nonlinear photonic lattices have been realized and discrete solitons observed [17]: using a suitable interference of two or more plane waves in a photosensitive material one could in principle create a non-translationally invariant photonic lattice. For Bose-Einstein condensates in optical lattices, discrete gap solitons and self-trapped states have been recently observed in linear arrays [18]: the control of the lattice shape in this system is realized properly superimposing the laser beams creating the optical lattices, as discussed in Ref. [19].

Here we address the issue of filtering the soliton propagation in the DNLSE, focusing on a particular class of inhomogeneous networks, built by adding a finite discrete network G^0 to a single site of a linear chain (see Fig. 1). We

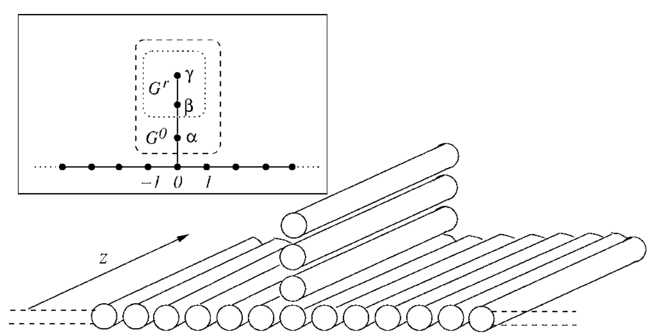

FIG. 1. An inhomogeneous system of coupled nonlinear waveguides extending in the z direction, obtained arranging the waveguides in a chain and attaching another finite chain in the waveguide 0. In the inset we plot the corresponding graph: each waveguide is a site of the graph, and two coupled waveguides are connected by a link. The attached graph G^0 is here a finite chain of length 3; G^r is obtained subtracting α from G^0.

refer to the physical setup of optical waveguides, however, the obtained results apply to the different physical settings described by the DNLSE. Of course, one can imagine and engineer a huge variety of network topologies; our aim is to exploit the interplay and competition of topology and nonlinearity and to show that the realistic topology obtained inserting properly chosen finite networks at a site of a chain can be used to build a novel class of filters for the soliton motion. By using a general criterion, derived within a linear approximation, relating the transmission properties of a soliton to the topology of the inserted network [12], we address the possibility of filtering the soliton propagation by suitably choosing the topology of the network. These analytical results are checked by a numerical study of the full nonlinear evolution.

The plan of the paper is the following: the next section is devoted to the introduction of the DNLSE on graphs and of the general approach used in the paper. In Sec. III we discuss the large-fast soliton regime and the general criterion for total transmission and reflection of solitons on inhomogeneous chains [12]. In Sec. IV we show that by using this general criterion one can choose the topology of the inserted network to realized filters and high-pass and low-pass devices. The transmission coefficient for the soliton propagation along the chain is computed for the proposed filters and high-pass and low-pass devices, and their efficiency discussed. In Sec. V we present our conclusions and future perspectives.

II. DNLSE ON GRAPHS

The DNLSE on a general network reads

$$i\frac{\partial E_n}{\partial z} = -\sum_{j=1}^{N}\beta_{n,j}E_j + \Lambda|E_n|^2E_n. \tag{1}$$

In the context of coupled optical waveguides $E_n(z)$ is the electric field in the nth waveguide ($n,j=1,\ldots,N$, with N being the total number of waveguides) and Λ is proportional to the nonlinear Kerr coefficient. When Λ has negative (positive) sign the medium has self-focusing (-defocusing) properties. The waveguides extend in the z direction, so one has that the soliton arising from the balance between discrete diffraction and nonlinearity are the discrete version of spatial optical solitons. In Eq. (1) the normalization is chosen to be $\Sigma_n|E_n(z)|^2=1$ and $\beta_{n,j}$ is proportional to the mode overlap of the electric fields of the waveguides n and j [3] and it is non-0 if n and j are nearest-neighbor waveguides, and 0 otherwise.

If N identical waveguides are arranged to form a chain, then Eq. (1) assumes the usual form [2],

$$i\frac{\partial E_n}{\partial z} = -\beta_c(E_{n+1}+E_{n-1}) + \Lambda|E_n|^2E_n, \tag{2}$$

where $\beta_{n,n\pm1}=\beta_c$. A (dispersion-managed) modulation $\beta(z)$ of the kinetic term has been discussed in Ref. [20]. If, on the other hand, the waveguides are arranged on the sites of a nontranslational invariant network, a space modulation of the kinetic term occurs, even if β does not depend on z.

The DNLSE on a homogeneous chain is not integrable [21]; nevertheless, solitonlike wave packets can be present [22,23], and the stability conditions of these solitonlike solutions may be discussed within a standard variational approach [23,24,14]. Let us consider, at $z=0$, a Gaussian wave packet, centered in ξ_0, with initial momentum k, and width γ_0. From the variational equations of motion, one sees that $k(z)=k$ is conserved. If $\Lambda>0$ ($\Lambda<0$), one can have a soliton solution only if $\cos k<0$ ($\cos k>0$). In the following, we assume $\Lambda>0$ and $\pi/2\leqslant k\leqslant\pi$ (positive velocities). A variational solitonlike solution is then obtained for $\gamma_0\gg1$ when Λ is given by [14]

$$\Lambda_{sol} \approx 4\beta_c\sqrt{\pi}\frac{|\cos k|}{\gamma_0}. \tag{3}$$

The stability of variational solutions can be numerically checked, showing that the shape of the solitons is preserved for long times. In the following, we use the term "solitons" to denote the solutions of the variational equations. We note that even if we are considering the large soliton regime ($\gamma\gg1$), our wave packet cannot be described as continuous, since its spatial modulation is of the same order of the lattice spacing. Therefore the packet is sensitive to the discrete nature of the support [e.g., the wave-packet velocity is proportional to $\sin(k)$ and not to k].

In this paper, we study the more complex situation, where an inhomogeneity is originated by a topological modification of the discrete lattice (see Fig. 1). We focus on the case where the graph G, describing the position of the waveguides, is obtained by attaching a finite graph G^0 to a single site of the chain [25]. In Ref. [26] we scattering of solitons through a topological perturbation linked at two sites of the linear chain has been considered, and in Ref. [27] an external Fano degree of freedom is coupled to several sites of the chain: here we will confine ourself to the situation in which a finite discrete network is attached at a single site of the chain, showing that, properly choosing the topology and the parameters of the waveguides composing the inserted network, one can realize different kinds of soliton filters. In Fig. 1 the attached graph G^0 is a chain of three sites (each site corresponds, of course, to a waveguide). We denote the generic waveguides with latin indices $i,j,\ldots$. A site of the graph will be denoted by an integer number or by a greek letter $\alpha,\beta,\ldots$, according as the corresponding waveguide belongs to the linear chain or to the graph G^0, respectively. A single link connects the waveguide 0 of the chain with the waveguide α of the graph G^0. Hereafter, we suppose that the waveguides of the infinite chain are identical, so that the coupling term between two generic neighboring sites of the infinite chain is set to the constant value β_c. The most general framework where the problem may be studied is provided by graph theory [28]: one can introduce the (generalized) adjacency matrix A^0 of G^0, defining $A^0_{i,j}=\beta_{i,j}$ when i and j are nearest-neighbors sites belonging to G^0, and 0 otherwise. Furthermore, we denote with G^r the graph obtained from G^0 by cutting the site α, and with A^r its generalized adjacency matrix: e.g., attaching a finite chain of three sites (α, β, and γ, see the inset of Fig. 1), to 0, G^r is a chain of length two having two sites (β, and γ) and A^r is a 2×2 matrix having vanishing diagonal elements, and $\beta_{\beta,\gamma}$ as nondiagonal terms. An energy level of G^0 is defined as an eigenvalue of the adjacency matrix A^0, and similarly an energy level of G^r is an eigenvalue of A^r.

The scattering of a soliton through this topological perturbation has been numerically studied in the following way. At $z=0$, we consider a Gaussian soliton, far left from 0 (i.e., $\xi_0<0$), moving towards $n=0$ [$\sin(k)>0$], and with a width γ_0 related to the nonlinear coefficient Λ through Eq. (3); i.e., we set $E_n(z=0)=\mathcal{K}\exp[-(n-\xi_0)^2/\gamma_0^2+ik(n-\xi_0)]$, where $\mathcal{K}$ is just a normalization factor. We numerically evaluate the nonlinear evolution of the electric field $E_n(z)$ at $z\neq 0$ from Eq. (1). The center of mass is defined as $\xi(z)=\Sigma_n n|E_n(z)|^2$ and the group velocity is $v=d\xi/dz$. For a soliton one has $v\approx 2\beta_c\sin k$ and, when $z=z_s\approx|\xi_0|/2\beta_c\sin(k)$, the soliton scatters through the finite graph G^0. At a position z well after the soliton scattering ($z\gg z_s$), we evaluate the reflection and transmission coefficients $\mathcal{R}$ and $\mathcal{T}$ using the formulas: $\mathcal{R}=\Sigma_{n<0}|E_n(z)|^2$ and $\mathcal{T}=\Sigma_{n>0}|E_n(z)|^2$.

III. A GENERAL RESULT FOR LARGE FAST SOLITONS

Let us introduce an important class of soliton solutions (to which we refer as large-fast solitons), whose scattering through a topological inhomogeneity can be analytically studied using a linear approximation [12]. The interaction between a soliton on a chain and a defect is characterized by two length scales [29]: the length of the soliton-defect interaction $z_{int}=\gamma_0/2\beta_c\sin k$ and the soliton dispersion space (i.e., the length scale in which the wave packet will spread in the absence of interaction) $z_{disp}=\gamma_0/[8\beta_c\sin(1/2\gamma_0)\cos k]$. When $\gamma_0\gg 1$ and $z_{int}\ll z_{disp}$, the soliton is very large with respect to the defect dimensions and it can be considered as a set of noninteracting plane waves while it experiences scattering by the graph. We extend this analysis to the scattering of a soliton through the attached graph G^0 and we compute the soliton transmission by considering, in the linear regime, the transport coefficients of a plane wave across the topological defect. Afterwords, we compares the analytical findings with a numerical solution of Eq. (1), namely with the reflection and transmission coefficients. We notice that for large-fast solitons $\mathcal{R}+\mathcal{T}\approx 1$, and no soliton trapping on the topological impurity occurs. We also point out that, even if the equation used to compute the transport coefficients is linear, the nonlinearity still plays a role: it sustains the soliton shape during its propagation (see Fig. 2). The use of the linear approximation for the analysis of the interaction of a fast soliton with a local defect in the continuous nonlinear Schrödinger equation has been reported in Ref. [30], while in Ref. [31] the transmission properties of narrow solitons (for which the linear approximation does not hold) in the DNLSE with a local defect were studied both numerically and with analytical techniques. Notice that for Bose-Einstein condensates in optical lattices the typical solitons have a spatial width γ larger than the lattice spacing a, e.g., $\gamma/a\sim 20$ in Ref. [18], so that large-fast solitons appear to be an appropriate approximation.

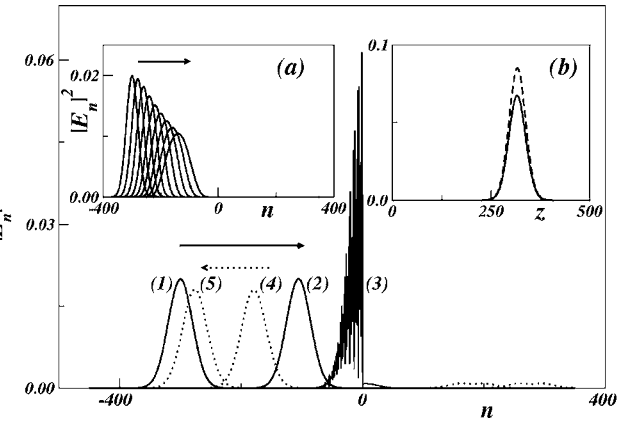

FIG. 2. Soliton propagation obtained from Eq. (1) [for momentum $k=1.8$, width $\gamma_0=40$, and Λ given by Eq. (3)] through a linear chains of two sites (α and β) with coupling term $\beta_{0,\alpha}=\beta_c$ and $\beta_{\alpha,\beta}=\beta_c/2$. The soliton profile is plotted for $z=0$, 200, 300, 500, 600 corresponding respectively to (1)–(5). The predicted reflection coefficient is close to 1 (see Fig. 3). Inset (a): wave-packet evolution for initial momentum $k=0.2$, showing that the wave packet spreads before to hit the topological defect. Inset (b): densities $|E_\alpha|^2$ (dashed line) and in G^0 vs z for $k_0=1.8$.

In the large-fast soliton regime, the momenta for perfect reflection and transmission are completely determined by the spectral properties of the graph G^0 [12]. The linear eigenvalue equation to investigate is $-\Sigma_m A_{n,m}E_m=\mu E_n$, where $A_{i,j}=\beta_{i,j}$ is the generalized adjacency matrix of the whole network. The momenta k corresponding to perfect reflection [$\mathcal{R}(k)=1$] and to perfect transmission [$\mathcal{T}(k)=1$] of a plane wave are determined by imposing the continuity at sites 0 and α. One obtains $\mathcal{R}=1$ if $2\beta_c\cos k$ coincides with an energy level of G^0, while $\mathcal{T}=1$ if $2\beta_c\cos k$ is an energy level of the reduced graph G^r [12]. This argument can be easily extended to the situation where p identical graphs G^0 are attached to 0.

IV. TOPOLOGICAL FILTERS

The general result previously stated allows for an identification of the graph G^0 selecting the transmission (or the reflection) of a particular (quasi)momentum k (a filter). A transmission filter can be obtained by inserting a finite chain of three sites (α, β, and γ; see Fig. 3). The coupling terms between 0 and α is set to the constant value β_c. The matrix A^0 is then given by

$$A^0=\begin{pmatrix}0 & \beta_1 & 0\\ \beta_1 & 0 & \beta_2\\ 0 & \beta_2 & 0\end{pmatrix}, \tag{4}$$

where $\beta_1\equiv\beta_{\alpha,\beta}$ is the coupling term between the waveguides α and β and $\beta_2\equiv\beta_{\beta,\gamma}$ is the coupling term between the waveguides β and γ. The eigenvalues of the matrix A^0 are $\lambda_1=0$ and $\lambda_{2\pm}=\pm\sqrt{\beta_1^2+\beta_2^2}$. According to the previously discussed general result, the values of momentum $k_{1,2}^{(R)}$ for which one has resonant reflection ($\mathcal{R}=1$) are $k_1^{(R)}=\pi/2$ and $k_2^{(R)}$ defined by

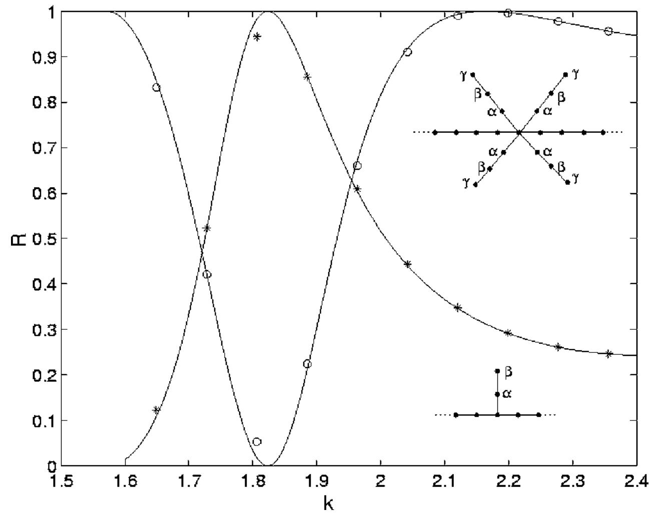

FIG. 3. The reflection coefficient vs the momentum k for a transmission filter and a reflection filter. A transmission filter can be obtained by inserting on a site of an infinite chain p linear chains of three sites (α, β, and γ). The coupling term between two generic waveguides of the chain is β_c. Furthermore, $\beta_{0,\alpha}=\beta_{\alpha,\beta}=\beta_c$ and $\beta_{\beta,\gamma}=\beta_c/2$. A reflection filter can be obtained inserting a linear chains of two sites (α and β) with coupling term between $\beta_{0,\alpha}=\beta_c$ and $\beta_{\alpha,\beta}=\beta_c/2$. Stars and circles are numerical results of Eq. (1) obtained, respectively, for the reflection filter and for the transmission filter (and $p=4$). The reflected and the transmitted momentum is ≈ 1.8. Solid lines corresponds to the analytical prediction (13).

$$\cos k_2^{(R)} = -\frac{\sqrt{\beta_1^2+\beta_2^2}}{2\beta_c} \tag{5}$$

(since we are choosing $\cos k<0$). With $\beta_1^2+\beta_2^2 \geqslant 4\beta_c^2$, the only momentum of perfect reflection is $k^{(R)}=\pi/2$, apart from $k=\pi$, which always corresponds to full reflection, since the soliton velocity ($\propto \sin k$) is zero. Furthermore, the momentum of perfect transmission is determined by diagonalizing the reduced adjacency matrix A^r which is given by

$$A^r = \begin{pmatrix} 0 & \beta_2 \\ \beta_2 & 0 \end{pmatrix} \tag{6}$$

whose eigenvalues are $\pm\beta_2$. Therefore the selected transmitted momentum is $k^{(T)}$, defined by

$$\cos k^{(T)} = -\frac{\beta_2}{2\beta_c}. \tag{7}$$

Then, one sees that if $\beta_2>2\beta_c$, it is not possible to realize a transmission filter with the discussed inserted network. When $\beta_2<2\beta_c$ one has a transmission filter: indeed the reflection coefficient $\mathcal{R}$ equals 1 at $k=\pi/2$, and reaches the value 0 at $k=k^{(T)}$ given by Eq. (7). If the condition $\beta_1^2+\beta_2^2<4\beta_c^2$ is also satisfied, then $\mathcal{R}$ increases until it reaches the value 1 at $k=k_2^{(R)}$ given by Eq. (5) (notice that $k_2^{(R)}>k^{(T)}$). For $k>k_2^{(R)}$, $\mathcal{R}$ decreases till a minimum value $\neq 0$ and, for a further increase of k, $\mathcal{R}$ increases and it reaches 1 at $k=\pi$. If the condition $\beta_1^2+\beta_2^2<4\beta_c^2$ is not satisfied, then increasing k (with $k>k_2^{(R)}$), $\mathcal{R}$ monotonically increases and it reaches 1 at $k=\pi$.

As previously discussed, the value of the perfect transmitted momentum does not change if p identical chains are attached in 0, but in this case the filter becomes more efficient. In the linear approximation, valid for large-fast solitons, the eigenvalue equation to investigate is

$$-\sum_n \beta_{m,n}E_n = \mu E_m \tag{8}$$

(here m and n are generic sites of the network). The solution corresponding to a plane wave coming from the left is on the sites of the chain $E_j=ae^{ikj}+be^{-ikj}$ for $j<0$ and $E_j=ce^{ikj}$ for $j>0$, so that $\mu=-2\beta_c\cos k$. The reflection coefficient is given by $\mathcal{R}=|b/a|^2$ and the transmission coefficient by $\mathcal{T}=|c/a|^2$. Denoting con $E_{\alpha,\beta,\gamma}$ the electric field in the waveguides α, β, γ, when p identical chains are inserted in 0 Eq. (8) for 0, α, β, and γ yields, respectively,

$$-\beta_c(ae^{-ik}+be^{ik}+ce^{ik}+p\psi_\alpha) = \mu(a+b), \tag{9}$$

$$-\beta_c(a+b) - \beta_1\psi_\beta = \mu\psi_\alpha, \tag{10}$$

$$-\beta_1\psi_\alpha - \beta_2\psi_\gamma = \mu\psi_\beta, \tag{11}$$

and

$$-\beta_2\psi_\beta = \mu\psi_\gamma. \tag{12}$$

Solving for b/a, Eqs. (9)–(12), together with the condition $a+b=c$, gives

$$\frac{1}{\mathcal{R}} = 1 + \frac{4\sin^2(2k)}{p^2}\left(1+\frac{\beta_1^2}{\beta_2^2-2\beta_c^2[1+\cos(2k)]}\right)^2. \tag{13}$$

One also gets $\mathcal{T}=1-\mathcal{R}$. Equation (13) confirms the qualitative behavior for the reflection coefficient previously discussed: indeed $\mathcal{R}(k=\pi/2)=1$ and $\mathcal{R}(k=\pi)=1$. Moreover, from Eq. (5), one has $\cos(2k_2^{(R)})=(\beta_1^2+\beta_2^2)/2\beta_c^2-1$, giving $\mathcal{R}(k=k_2^{(R)})=1$. Similarly, from Eq. (7), one has $\cos(2k^{(T)})=\beta_2^2/2\beta_c^2-1$, giving $\mathcal{R}(k=k^{(T)})=0$.

From Eq. (13) one can also estimate the filter efficiency: indeed, around $k=k^{(T)}$, one has $\mathcal{R}\approx\frac{1}{2}C(k-k^{(T)})^2$, where $C=\left(\frac{\partial^2\mathcal{R}}{\partial k^2}\right)_{k=k^{(T)}}$. Then, one can estimate what is the maximum value of the momentum shift $\Delta k=k-k^{(T)}$ for which one has $\mathcal{R}\leqslant\epsilon$, where ϵ quantifies the efficiency of the transmission filter. From Eq. (13) one gets

$$C = 8p^2\left(\frac{\beta_c}{\beta_1}\right)^4: \tag{14}$$

it follows that the maximum momentum deviation Δk_{MAX} is given by

$$\Delta k_{MAX} = \sqrt{\frac{2\epsilon}{C}} = \left(\frac{\beta_1}{\beta_c}\right)^2\frac{\sqrt{\epsilon}}{2p}. \tag{15}$$

From Eq. (15) one sees that one can improve the filter selectivity simply increasing the number p of inserted chains with length 3; at variance, the selected transmitted momentum $k^{(T)}$ does not depend on p.

To implement a reflection filter, i.e., a filter selecting only a particular $k^{(R)}$ for which $\mathcal{T}(k^{(R)})=0$, one can add, e.g., p finite chains with $L=2$ sites inserted in 0. As before, we label by α and β the two sites of the added chain. We suppose that the coupling term between 0 and α is β_c, while the coupling term between α and β is $\beta_{\alpha,\beta}\equiv\beta_1$. The matrix A^0 is then given by

$$A^0=\begin{pmatrix}0 & \beta_1\\ \beta_1 & 0\end{pmatrix}: \tag{16}$$

the eigenvalues of the matrix A^0 are $\lambda_\pm=\pm\beta_1$. The only value of the momentum $k^{(R)}$ for which one has resonant reflection $(\mathcal{R}=1)$ is defined by

$$\cos k^{(R)}=-\frac{\beta_1}{2\beta_c}. \tag{17}$$

With $\beta_1<2\beta_c$, a momentum perfectly reflected exists, apart from $k=\pi$, which always corresponds to full reflection. The matrix A^r reduces to a single element $(A^r=0)$ and the value at which $\mathcal{T}=1$ is only $k=\pi/2$.

From Eq. (17) one sees that when $\beta_1<2\beta_c$ one has a reflection filter: indeed the reflection coefficient $\mathcal{R}$ equals 0 at $k=\pi/2$, and reaches the value 1 at $k=k^{(R)}$ given by Eq. (17). For $k>k^{(R)}$, $\mathcal{R}$ decreases till a minimum value $\neq 0$ and, for a further increase of k, $\mathcal{R}$ increases and it reaches 1 at $k=\pi$.

The corresponding analytical expression for $\mathcal{R}$ is obtained from Eq. (13) setting $\beta_2=0$:

$$\frac{1}{\mathcal{R}}=1+\frac{4\sin^2(2k)}{p^2}\left(1-\frac{\beta_1^2}{2\beta_c^2[1+\cos(2k)]}\right)^2. \tag{18}$$

Equation (13) confirms the qualitative behavior for the reflection coefficient previously discussed: indeed $\lim_{k\to\pi/2}\mathcal{R}=0$ and $\mathcal{R}(k=\pi)=1$. Moreover, from Eq. (17), one has $\cos(2k^{(R)})=\beta_1^2/2\beta_c^2-1$, giving $\mathcal{R}(k=k^{(R)})=1$. The filter efficiency can be estimated by observing that around $k=k^{(R)}$ one has $\mathcal{R}\approx 1+\frac{1}{2}C(k-k^{(R)})^2$, where $C=\left(\frac{\partial^2\mathcal{R}}{\partial k^2}\right)_{k=k^{(R)}}$. We now compute the maximum value of the momentum shift $\Delta k=k-k^{(R)}$ for which one has $\mathcal{R}\geqslant 1-\epsilon$ (i.e., $\mathcal{T}\leqslant\epsilon$), where ϵ quantifies the efficiency of the reflection filter. From Eq. (18) one gets

$$C=-\frac{128}{p^2}\left(1-\frac{\beta_1^2}{4\beta_c^2}\right)^2: \tag{19}$$

it follows that the maximum momentum deviation Δk_{MAX} is given by

$$\Delta k_{MAX}=\sqrt{\frac{2\epsilon}{|C|}}=\frac{p\sqrt{\epsilon}}{4}\sqrt{\frac{\beta_c^2}{4\beta_c^2-\beta_1^2}}. \tag{20}$$

From Eq. (20) one sees that increasing the number p of inserted chains with length 2, the filter selectivity make worse; we notice that this result is the opposite of what happens for the transmission filter previously discussed [compare with Eq. (15)]. Also for the reflection filter, the selected reflected momentum $k^{(R)}$ does not depend on p.

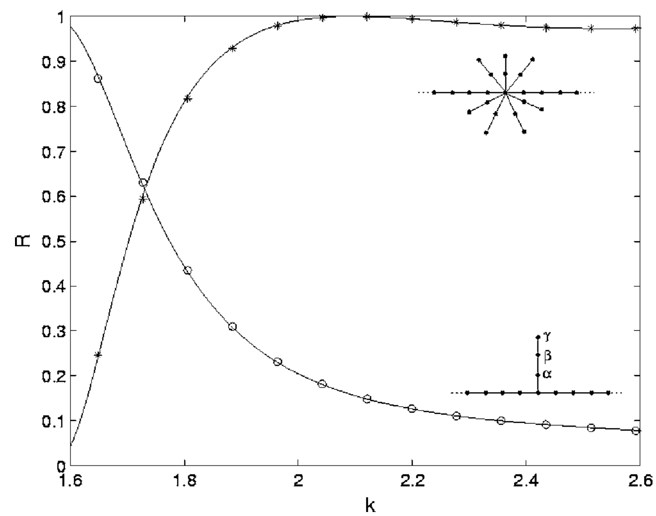

FIG. 4. The reflection coefficient vs the momentum k for a low-pass filter and a high-pass filter. Stars and circles are numerical results of Eq. (1) obtained for the for the high-pass (and $p=7$) and for the low-pass. Solid lines corresponds to the analytical prediction (13).

The transmission properties of the transmission and reflection filters are shown in Fig. 3. The analytical results are compared with numerical findings: as discussed in Sec. II, in the numerical simulations we consider as initial condition $(z=0)$ a Gaussian soliton, far left from 0, moving towards $n=0$ $(\sin k>0)$, and with a width γ_0 related to the nonlinear coefficient Λ through Eq. (3): in the simulations we used $\gamma_0=40$. At a position z well after the soliton scattering on the topological impurity, we evaluate the reflection and transmission coefficients $\mathcal{R}$ and $\mathcal{T}$ using the formulas: $\mathcal{R}=\Sigma_{n<0}|E_n(z)|^2$ and $\mathcal{T}=\Sigma_{n>0}|E_n(z)|^2$. Figure 3 evidences the good agreement between numerical findings and analytical results, and the good efficiency of the discussed filters.

Finally, using the obtained results, one is also able to realize a high-pass filter which allows the transmission of high velocity solitons (i.e., k close to $\pi/2$ and high soliton velocities) and a low-pass filter transmitting the low velocity solitons (i.e., k close to π and low velocities). A possible network realizing a high-pass filter is obtained by attaching p finite chains with length 2 and all coupling terms fixed to β_c. The analytical expression for the reflection coefficients $\mathcal{R}$ is obtained by setting $\beta_1=\beta_c$ and $\beta_2=0$ in Eq. (13). This structure acts as a high-pass, with a cutoff momentum (and an efficiency) depending on p. In Fig. 4, $\mathcal{R}$ vs k is plotted for $p=7$. On the other hand, the low-pass effect can be obtained with a linear chain of three sites (α, β, and γ) inserted in 0 and coupling terms β_c, β_1, β_2, respectively between 0 and α, α and β, β and γ. If $\beta_2=2\beta_c$, then $\mathcal{T}=0$ for $k=\pi/2$ and $\lim_{k\to\pi}\mathcal{R}=0$ [although $\mathcal{R}(\pi)=1$]. Therefore one has a low-pass with the cutoff momenta depending on β_1 (see Fig. 4, where the values $\beta_1=\beta_c$ and $\beta_2=2\beta_c$ are used).

V. CONCLUSIONS

In conclusion, we studied the discrete nonlinear Schrödinger equation on an inhomogeneous structure, obtained attaching a finite graph G^0 to a site of a linear chain. We showed that one can determine the topology and the parameters of G^0, giving reflection and transmission filters, or low-pass/high-pass filters. Our results apply to solitons with a length scale much larger than the typical distance between waveguides: as a promising future work to be done, we mention that the study of solitons and breathers at the same scale of the interguide distance should give an even richer variety of phenomena arising from the interplay between discreteness, nonlinearity and topology.

The results obtained show the remarkable influence of topology on nonlinear dynamics, and apply in general to soliton propagation in discrete networks whose shape is controllable. As these results suggest, we feel that it is now both timely and highly desirable to develop the investigation of nonlinear models on general inhomogeneous networks, since one should expect new and interesting phenomena arising from the interplay between nonlinearity and topology.

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